Pub. Mat. UAB
Vol. 29 Nº 2-3 Nov. 1985

HOMOLOGY OF ONE-RELATOR GROUPS WITH TWISTED COEFFICIENTS

Grzegorz Gromadzki

1. Introduction.

It is well known that for a torsion-free, one-relator group $G = (x_1, \ldots, x_n : r)$ we have $H_2(G,Z) = Z$ if and only if $r \in [F,F]$. Such groups have received recently some attention. Horadam has found some invariants of such groups [5,6] and Ratcliffe has characterized those of such groups that are oriented trivial Poincare duality groups of dimension 2 [10], see also [11].

One is naturally led to ask for a similar characterization of those torsion-free, one-relator groups for which there exists a G-module structure Z_α on integers such that $H_2(G,Z_\alpha) = Z$. Following the tradition we will refer to any such G-module structure as to twisted coefficients. The simple example of two groups $G = (x,y : x^2y^4[y,x])$, $H = (x,y : x^2y^4[x,y])$, the first of which has the abovementioned property while the second does not, shows that such characterization seems to be nontrivial. The lemma 1.1 stated below was proved having in mind this problem but its consequences are more far-reaching. In order to state it let F be a free group on free generators $\{x_i : i \in I\}$ and for any homomorphism $\alpha : F \longrightarrow \{\pm 1\}$ consider the normal subgroup $[F,F]_\alpha$ of F generated by the following elements:

(i) x_i^2 when $\alpha(x_i) = -1$,

(ii) $x_i x_j x_i^{-1} x_j^{-\alpha(x_i)}$ when $\alpha(x_j) = +1$,

(iii) $(x_i x_j x_k)^2$ when $\alpha(x_i) = \alpha(x_j) = \alpha(x_k) = -1$ and i,j,k are pairwise distinct.

Moreover let IF be the augmentation ideal of ZF and let $I_\alpha F = \mathrm{Ker}(\tilde{\alpha} : ZF \longrightarrow Z,\ \tilde{\alpha}(\Sigma\, n_f f) = \Sigma\, n_f \alpha(f))$.

Lemma 1.1. For any element f of F following statements are equivalent

(i) $f \in [F,F]_\alpha$,

(ii) $f - 1 \in I_\alpha F \cdot IF$,

(iii) $\partial f / \partial x_i \in I_\alpha F$ for all $i \in I$.

Using the Lyndon resolution we obtain that for a group $G = (x_1,\dots,x_n : r)$ the second homology group of G with twisted coefficients $\alpha : G \longrightarrow \mathrm{Aut}\, Z = \{\pm 1\}$ is equal to Z if an only if $r \in [F,F]_{\alpha'}$, where α' is determined by α

and by the given presentation of G (Proposition 3.1(i)). This gives us the characterization we have looked for. As a corollary we obtain also that $[F,F]_\alpha$ is equal to the subgroup of F generated by elements w^2, $uvu^{-1}v^{-\alpha(u)}$, where $\alpha(w) = -1, \alpha(v) = +1$.

Note that the group $[F,F]_\alpha$ has some topological flavour. Namely, generators of type (i) arise from a projective plane, generators of type (ii) from a torus or a Klein bottle according as $\alpha(x_i) = 1$ or -1 and finally note that one can exclude generators of type (iii) by choosing an appropriate set of free generators of F. Thus it is natural that the cap product will be useful tool for studying one-relator groups with a relator in $[F,F]_\alpha$. We study the cap product for such groups in section 3.

As was quoted earlier Horadam has found invariants of torsion-free, one-relator groups with nontrivial second integral homology. Using computations of section 3 we find in section 4 a certain invariant of torsion-free, one-relator groups with a relator in $[F,F]_\alpha$ for some α (Theorem A). Our invariant allows us to distinguish certain groups for which Horadam invariants were insufficient. We give some examples at the end of this section.

Finally in section 5 we study those one-relator groups which satisfy trivial Poincare duality (TPD-groups). Our characterization (Theorem B) extends those of Ratcliffe and Fenn -

Sjerve obtained in the case of oriented TPD-groups.

This paper is a part of my Ph.D. thesis written at the Mathematical institute of the Polish Acadamy of Sciences under the supervision of Professor Stanislaw Balcerzyk. I wish to thank him for the help in the preparation of this paper. I would like to thank also Dr. Tomasz Bortnik for numerous helpful discussions.

2. Proof of the main lemma.

Here we prove Lemma 1.1 stated in the previous section. Note first that it is sufficient to prove the lemma in the case of F finitely generated and α non-trivial. Thus let $I = \{1,\ldots,n\}$. For notational convenience assume that $\alpha(x_1)=\ldots=\alpha(x_k) = -1$ and $\alpha(x_{k+1}) = \ldots = \alpha(x_n) = +1$, $0 < k \leqslant n$.

Implication (iii) $\Rightarrow$ (ii) is an immediate consequence of the fundamental Fox formula $f-1 = \sum_{i=1}^{n} \partial f/\partial x_i (x_i-1)$ [2]. (ii) $\Rightarrow$ (iii) follows from this formula and from the freeness of IF as F-module. In order to prove (i) $\Rightarrow$ (iii) note first that any generator of $[F,F]_\alpha$ satisfies (iii). Thus the assertion follows from the basic properties of Fox derivatives. Thus we have to prove implication (iii) $\Rightarrow$ (i) only. We will prove it using following technical lemmas (hereafter we use left-hand notations for commutators and conjugation, $\bar{x}$ denotes the inverse of x and $x \equiv y$ means $x \equiv y \bmod [F,F]_\alpha$ unless otherwise stated).

Lemma 2.1. Let $x,y,z \in \{x_1,\dots,x_n\}$ and let $\epsilon,\delta,\eta = \pm 1$. Then

(a) $[x^\epsilon, y^\delta] \equiv 1$ if $\alpha(x) = \alpha(y) = +1$,

(b) $(x^\epsilon y^\delta z^\eta)^2 \equiv 1$ if $\alpha(x) = \alpha(y) = \alpha(z) = -1$,

(c) $x^\epsilon y^\delta \bar{x}^\epsilon \bar{y}^{\delta\alpha(x)} \equiv 1$ if $\alpha(y) = +1$.

Proof. Case by case straightforward verification.

Lemma 2.2. Let $x,y,z,w \in \{x_1,\dots,x_n\}$, $f \in F$ and $\epsilon,\delta = \pm 1$.

Then

(a) $(x^{2\epsilon})^f \equiv x^{2\epsilon\alpha(f)}$,

(b) $[x,y]^f \equiv x^{2\alpha(f)}$ if $\alpha(x) = +1$, $\alpha(y) = -1$,

(c) $[x,y]^f \equiv [x,y]^{\alpha(f)}$ if $\alpha(x) = \alpha(y) = -1$,

(d) $[x,y][z,w] \equiv [z,w][x,y]$ if $\alpha(x) = \alpha(y) = -1$

(e) $x^{2\epsilon}y^{2\delta} \equiv y^{2\delta}x^{2\epsilon}$ if $\alpha(x) = \alpha(y) = +1$,

(f) $[y,x][x,z] \equiv [y,z]$ if $\alpha(x) = \alpha(y) = \alpha(z) = -1$.

Proof. (a) Let $\alpha(x) = +1$. It is easy to check that for any $v \in \{x_1,\dots,x_n\}$, $(x^2)^v = x^2[\bar{x},v]^{\bar{x}}[\bar{x},v] = \bar{x}^2(vx\bar{v}x)^{x^2}(vx\bar{v}x)^x$. By substitution of x^ϵ or v^δ in the place of x or v we obtain similar formulas for $(x^{2\epsilon})^{v^\delta}$. Using the first or the second form according as $\alpha(v) = +1$ or -1 we obtain by the previous lemma the assertion in the case that f is of length 1. Thus in order to finish the proof of the case (a) it is sufficient to use the induction on the length of f. The case $\alpha(x) = -1$ is trivial by the definition of $[F,F]_\alpha$.

(b) Note that $[x,y]^f = (x^2)^f(yx\bar{y}x)^{-f}$. Hence (b) is the consequence of (a).

(c) One can check that for any $v \in \{x_1,\dots,x_n\}$ we have

$[x,y]^v = [x,y](\bar{x}vxv)^{yx\bar{y}}(\bar{y}\bar{v}y\bar{v})^{yx}(xv\bar{x}v)^y(y\bar{v}\bar{y}\bar{v}) =$

$[x,y]^{-1}((\bar{x}\bar{y}v)^2)^{xy}(xy\bar{v})^2(\bar{y}^2)^v(x^2)^{vy\bar{x}y}((y\bar{x}y)^2)^v(\bar{y}^2)^v$. Now we obtain (c) as in the proof of (a).

(d) is the corollary of (c).

(e) is obvious.

(f) follows from the formula $[y,x][x,z] =$

$[y,z]((y\bar{z}x)^2)^z(\bar{y}^2)^{z\bar{x}z}(z^2)^{z\bar{x}}(\bar{x}^2)^z$ combined with Lemma 2.1.

Now we are ready to prove the outstanding implication (iii) $\Rightarrow$ (i) of Lemma 1.1. Let $f \in F$ satisfy (iii). It is easy to see that $f \in F^2[F,F]$ in this case. Thus f can be written in the following form

$$f = \prod_{i=1}^{n} x_i^{n_i} \prod_{j=1}^{N_1} [x_{i_j^1}, x_{i_j^2}]^{\epsilon_j f_j^1} \prod_{j=1}^{N_2} [x_{i_j^3}, x_{i_j^4}]^{f_j^2} \prod_{j=1}^{N_3} [x_{i_j^5}, x_{i_j^6}]^{f_j^3}$$

where $i_j^1, i_j^3, i_j^4 \in \{1,\dots,k\}$, $i_j^2, i_j^5, i_j^6 \in \{k+1,\dots,n\}$, $f_j^1, f_j^2, f_j^3 \in F$, $\epsilon_j = \pm 1$ and n_i are even integers.

Using N_1-times (b), N_2-times (c) then (e) of Lemma 2.2 we obtain that

$$f \equiv \prod_{i=k+1}^{n} x_i^{m_i} \prod_{j=1}^{N_2} [x_{i_j^3}, x_{i_j^4}]^{\eta_j},$$

where m_i are even integers and $\eta_j = \pm 1$.

Now we see that for $i = k+1,\dots,n$ $\tilde{\alpha}(\partial f/\partial x_i) = m_i$. Hence all m_i must be zero and thus

$$f \equiv \prod_{j=1}^{N_2} [x_{i_j^3}, x_{i_j^4}]^{\eta_j}$$

Using (d) of Lemma 2.2 we infer that

$$f \equiv \prod_{j=1}^{M_1} [x_{k_j^1}, x_k] \prod_{j=1}^{M_2} [x_k, x_{k_j^2}] \prod_{j=1}^{M_3} [x_{k_j^3}, x_{k_j^4}]$$

where $k_j^1, k_j^2, k_j^3, k_j^4 \in \{1,\ldots,k-1\}$. So $\tilde{\alpha}(\partial f/\partial x_k) = 2(M_2 - M_1)$. Therefore by (d) and (f) of Lemma 2.2 we have that

$$f \equiv \prod_{j=1}^{K} [x_{l_j^1}, x_{l_j^2}],$$

where $l_j^1, l_j^2 \in \{1,\ldots,k-1\}$.

Repeating the above procedure for generators $x_{k-1},\ldots,x_1$ we obtain that $f \equiv 1$. Hereby the lemma is proved.

As the immediate consequence of Lemma 1.1 we have

<u>Corollary</u> 2.3. The subgroup $[F,F]_\alpha$ coincides with the subgroup of F generated by the set $\{w^2, uv\bar{u}\bar{v}^{\alpha(u)} : u,v,w \in F$ and $\alpha(w) = -1, \alpha(v) = +1\}$.

3. <u>Cap product for torsion-free, one-relator groups</u>.

Here we use the Lyndon resolution in order to find an explicit formula for the cap product of a torsion-free, one-relator group with a relator in $[F,F]_\alpha$.

The most convenient resolution for the description of the cap product is the standard bar resolution $\mathfrak{B}$. Let A and B be G-modules. Then the cap product is a family of homomorphisms

$\cap : H_n(G,A)\otimes H^k(G,B) \longrightarrow H_{n-k}(G,A\otimes B)$ induced by the morphisms of chain complexes $\cap : (A \underset{G}{\otimes} S_n)\otimes \mathrm{Hom}_G(S_k,B) \longrightarrow (A\otimes B)\underset{G}{\otimes} S_{n-k}$ given by $a\otimes(g_0,\dots,g_n)\cap f = a\otimes f(g_0,\dots,g_k)\otimes(g_k,\dots,g_n)$.

For a torsion-free, one-relator group $G = (x_1,\dots,x_n : r)$ we have the useful Lyndon resolution [7]:

$$\mathbb{L} : 0 \longrightarrow ZG \xrightarrow{d_2} ZG^n \xrightarrow{d_1} ZG \xrightarrow{\epsilon} Z \longrightarrow 0$$

where ϵ is the augmentation homomorphism, $d_1(a_1,\dots,a_n) = \sum_{i=1}^{n} a_i.(p(x_i) - 1)$, $d_2(a)=(a.p(\partial r/\partial x_1),\dots,a.p(\partial r/\partial x_n))$ ($p:F\to G$ is the canonical projection given by the presentation of G).

Using this resolution for a torsion-free, one-relator group $G = (x_1,\dots,x_n:r)$ and Lemma 1.1 we get at once

<u>Proposition</u> 3.1. (i) $H_2(G,Z_\alpha) = Z$ if and only if $r\in[F,F]_{\alpha'}$, otherwise $H_2(G,Z_\alpha) = 0$, where α' is determined by α and by the given presentation of G.

(ii) Let $r\in[F,F]_{\alpha'}$. Then $H_1(G,Z_\alpha) = Z^n$ or Z^{n-1} according as α is trivial or not.

Hereafter for a given presentation of G we use the same notation for the homomorphism $G \longrightarrow \{\pm 1\}$ and for the corresponding homomorphism $F \longrightarrow \{\pm 1\}$.

There are chain transformations $\phi : \mathbb{L} \longrightarrow \mathbb{S}$ and $\psi : \mathbb{S} \longrightarrow \mathbb{L}$ extending the homomorphism id_Z that induce isomorphisms on homology and cohomology groups. We choose ϕ as follows

$$\phi_0 = id_{ZG},$$

$$\phi_1(a_1,\ldots,a_n) = \sum_{i=1}^{n} a_i.(1,p(x_i)),$$

$\phi_2(1) = \sum_{i=1}^{n} \sum_{j=1}^{n_i} \epsilon_j^i r_j^i (1,p(x_i),p(\bar{r}_j^i))$, where $\partial r/\partial x_i = \sum_{j=1}^{n_i} \epsilon_j^i r_j^i$

$\phi_k = 0$ for $k > 2$

and let for ψ

$$\psi_0 = id_{ZG},$$

$$\psi_1(1,g) = (p(\partial s(g)/\partial x_1),\ldots,p(\partial s(g)/\partial x_n)), \text{ where}$$

$s : G \longrightarrow F$ is a function satisfying $p.s = id_G$ and $s(p(x_i)) = x_i$ for $i = 1,\ldots,n$.

As far as we know Marciniak [8] was the first who use these transformations in study of one-relator groups.

Hereafter in this section we restrict our attention to torsion-free, one-relator groups with a relator in $[F,F]_\alpha$ for some α. Let $e_r^\alpha = 1 \otimes \phi_2(1) \in Z_\alpha \otimes_G Z[G^3]$. The following lemma is obvious.

<u>Lemma</u> 3.2. The element e_r^α is a cycle in $Z_\alpha \otimes_G \mathfrak{S}$ and its homology class e_G^α generates $H_2(G,Z_\alpha) = Z$.

The next lemma allows us to find an explicit formula for the cap product.

<u>Lemma</u> 3.3. The following diagrams are commutative

(i)

$$\begin{array}{ccc} \mathrm{Hom}_G(Z[G^3],Z) & \xrightarrow{e_r^\alpha \cap -} & Z_\alpha \otimes_G Z[G] \\ \big\downarrow \phi_2^* & & \big\uparrow \phi_0^* \\ \mathrm{Hom}_G(ZG,Z) & \xrightarrow{\eta} & Z_\alpha \otimes_G ZG \end{array}$$

where $\eta(f) = f(1) \otimes 1$,

(ii)

$$\begin{array}{ccc} \mathrm{Hom}_G(Z[G^2],Z) & \xrightarrow{e_r^\alpha \cap -} & Z_\alpha \otimes_G Z[G^2] \\ \big\uparrow \psi_1^* & & \big\downarrow \psi_{1*} \\ \mathrm{Hom}_G(ZG^n mZ) & & Z_\alpha \otimes_G ZG^n \\ \big\uparrow \cong & & \big\downarrow \cong \\ Z^n & \xrightarrow{A_r^\alpha} & Z^n \end{array},$$

where $(A_r^\alpha)_{i,j} = -\tilde{\alpha}(\partial^2 r/\partial x_j \partial x_i)$,

$$\begin{array}{ccc} \mathrm{Hom}_G(Z[G],Z) & \xrightarrow{e_r^\alpha \cap -} & Z_\alpha \otimes_G Z[G^3] \\ \big\downarrow \phi_0^* & & \big\uparrow \phi_{2*} \\ \mathrm{Hom}_G(ZG,Z) & \xrightarrow{\eta} & Z_\alpha \otimes_G ZG \end{array},$$

where we identify $Z_\alpha \otimes Z$ and Z_α.

<u>Prof</u>. The proofs of (i) and (iii) are trivial, we prove (ii) only. Let $f_k \in \mathrm{Hom}_G(ZG^n, Z)$ be homomorphism corresponding to the standard base vector $e_k \in Z^n$. Thus $f_k(\hat{e}_i) = \delta_{i,k}$, where $\{\hat{e}_i\}$ is the standard basis for ZG^n.

$$e_r^\alpha \cap \psi_1^* (f_k) = e_r^\alpha \cap f_k \psi_1 =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n_j} \epsilon_j^i \alpha(r_j^i) f_k \psi_1(1, p(x_i)) \otimes_G (p(x_i), p(\bar{r}_j^i)) =$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n_i} \epsilon_j^i \alpha(r_j^i) f_k(\hat{e}_i) \otimes_G p(x_i).(1, p(\bar{x}_i \bar{r}_j^i)) =$$

$$\sum_{j=1}^{n_k} \epsilon_j^k \alpha(x_k r_j^k) \otimes_G (1, p(\bar{x}_k \bar{r}_j^k)).$$

Since $r \in [F,F]_\alpha$ we have by Lemma 1.1 that $\widetilde{\alpha}(\partial r/\partial x_i) = 0$ for $i = 1,\dots,n$. Hence $\psi_{1*}(e_r^\alpha \cap \psi_1^*(f_k)) =$

$$\sum_{j=1}^{n_k} \epsilon_j^k \alpha(x_k r_j^k) \underset{G}{\otimes} (p(\partial(\bar{x}_k \bar{r}_j^k)/\partial x_1), \dots, p(\partial(\bar{x}_k \bar{r}_j^k)/\partial x_n)) =$$

$$\sum_{j=1}^{n_k} \epsilon_j^k \alpha(x_k r_j^k) \underset{G}{\otimes} (\sum_{i=1}^{n} (-p(\bar{x}_k \bar{r}_j^k) p(\partial r_j^k/\partial x_i) . \hat{e}_i - p(\bar{x}_k) . \hat{e}_k)) =$$

$$- \sum_{j=1}^{n_k} \sum_{i=1}^{n} \epsilon_j^k \alpha(x_k r_j^k) \alpha(\bar{x}_k \bar{r}_j^k) \widetilde{\alpha}(\partial r_j^k/\partial x_i) \underset{G}{\otimes} \hat{e}_i - \sum_{j=1}^{n_k} \epsilon_j^k \alpha(x_k r_j^k) \alpha(\bar{x}_k) \underset{G}{\otimes} \hat{e}_k.$$ But

$$\sum_{j=1}^{n_k} \epsilon_j^k \alpha(x_k r_j^k) \alpha(\bar{x}_k) \underset{G}{\otimes} \hat{e}_k = \sum_{j=1}^{n_k} \epsilon_j^k \alpha(r_j^k) \underset{G}{\otimes} \hat{e}_k = \widetilde{\alpha}(\partial r/\partial x_k) \underset{G}{\otimes} \hat{e}_k = 0$$

Therefore $\psi_{1*}(e_r^\alpha \cap \psi_1^*(f_k)) = - \sum_{i=1}^{n} \sum_{j=1}^{n_k} \epsilon_j^k \widetilde{\alpha}(\partial r_j^k/\partial x_i) \underset{G}{\otimes} \hat{e}_i =$

$$- \sum_{i=1}^{n} \widetilde{\alpha}(\partial^2 r/\partial x_k \partial x_i) \underset{G}{\otimes} \hat{e}_i = \sum_{i=1}^{n} (A_r^\alpha)_{i,k} \underset{G}{\otimes} \hat{e}_i \xrightarrow{\cong} \sum_{i=1}^{n} (A_r^\alpha)_{i,k} . e_i .$$

The following lemma can be proved using fundamental properties of the Fox derivatives.

<u>Lemma</u> 3.4. Let $s,t \in [F,F]_\alpha$, $f \in F$. Then

(i) $A_{st}^\alpha = A_s^\alpha + A_t^\alpha$,

(ii) $A_{\bar{s}}^\alpha = - A_s^\alpha$

(iii) $(A^{\alpha}_{fs\bar{f}})_{i,j} = \alpha(f)(A^{\alpha}_{s})_{i,j} - \epsilon(\partial s/\partial x_j)\widetilde{\alpha}(\partial f/\partial x_i)$.

From now till the end of this section we assume that $r \in [F,F]_{\alpha_0}$, where $\alpha_0(x_1) = -1$, $\alpha_0(x_2) = \ldots = \alpha_0(x_n) = +1$. In this case $[F,F]_{\alpha_0}$ is normally generated by the elements: x_1^2, $x_i x_j \bar{x}_i \bar{x}_j{}^{\alpha(x_i)}$, $i = 1,\ldots,n$, $j = 2,\ldots,n$

Lemma 3.5

$$(A^{\alpha_0}_{x_1^2})_{k,\ell} = -1 \quad \text{if} \quad k = \ell = 1,$$
$$= 0 \text{ otherwise.}$$

$$(A^{\alpha_0}_{x_1 x_i \bar{x}_1 x_i})_{k,\ell} = +1 \quad \text{if} \quad k = \ell = i$$
$$= -1 \quad \text{if} \quad k = i,\ \ell = 1 \text{ or } k = 1, \ell = i,$$
$$= 0 \quad \text{otherwise.}$$

$$(A^{\alpha_0}_{[x_i,x_j]})_{k,\ell} = -1 \quad \text{if} \quad k = i,\ \ell = j,$$
$$= +1 \quad \text{if} \quad k = j,\ \ell = i,$$
$$= 0 \text{ otherwise.}$$

Proof. Direct computation.

Lemma 3.6. Let $r \in [F,F]\alpha_0$. Then $-2(A^{\alpha_0}_r)_{1,i} = \epsilon(\partial r/\partial x_i)$ for $i = 1,\ldots,n$.

Proof. Let $r = \prod_{j=1}^{m} w_j r_j \bar{w}_j$, where $w_j \in F$ and r_j are generators of $[F,F]_{\alpha_0}$ or their inverses. Since $\epsilon(\partial r/\partial x_i) = \sum_{j=1}^{m} \epsilon(\partial r_j/\partial x_i)$

hence by 3.4 it is sufficient to show that for any generator r_k of $[F,F]_{\alpha_0}$ or for its inverse and for any $w \in F$ we have

$$2.(\alpha_0(w)\tilde{\alpha}_0(\partial^2 r_k/\partial x_i \partial x_1) + \tilde{\alpha}_0(\partial w/\partial x_1)\epsilon(\partial r_k/\partial x_i)) = \epsilon(\partial r_k/\partial x_i).$$

It can be done using two previous lemmas and easy

<u>Lemma</u> 3.7 Let $w \in F$. Then $\tilde{\alpha}_0(\partial w/\partial x_1) = 0$ if $\alpha_0(w) = +1$,
$= 1$ otherwise

Now we may summarize the results of this section.

<u>Proposition</u> 3.8 Let $G = (x_1,\ldots,x_n : r)$ be a torsion-free, one-relator group with $r \in [F,F]_{\alpha_0}$. Then

(i) homomorphism $e_G^{\alpha_0} \cap - : H^0(G,Z) \longrightarrow H_2(G,Z_{\alpha_0})$ is an isomorphism,

(ii) homomorphism $e_G^{\alpha_0} \cap - : H^2(G,Z) \longrightarrow H_0(G,Z_{\alpha_0})$ is an epimorphism $Z \longrightarrow Z/2Z$ or isomorphism $Z/2Z \longrightarrow Z/2Z$ according as $r \in [F,F]$ or not, moreover there are commutative diagrams with exact rows

(iii)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G,Z) & \longrightarrow & Z^n & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e_G^{\alpha_0}\cap -} & & \downarrow{\scriptstyle A_r^{\alpha_0}} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & H^1(G,Z_{\alpha_0}) & \longrightarrow & Z^n & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

if $r \notin [F,F]$,

(iv)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1(G,Z) & \longrightarrow & Z^n & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle e_G^{\alpha_0}} & & \downarrow{\scriptstyle A_r^{\alpha_0}} & & \downarrow & & \\ 0 & \longrightarrow & H^1(G,Z_{\alpha_0}) & \longrightarrow & Z^n & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

if $r \in [F,F]$.

Proposition 3.9. Let $G = (x_1,\dots,x_n : r)$ be a torsion-free group with $r \in [F,F]$. Then there is commutative diagram

$$\begin{tikzcd} H^1(G,Z) \arrow[r] \arrow[d, "e_G^{\epsilon} \cap -"'] & Z^n \arrow[d, "A_r^{\epsilon}"] \\ H^1(G,Z) \arrow[r] & Z^n \end{tikzcd}$$

in which horizontal maps are isomorphisms.

4. Invariants of one-relator groups.

Let $G = (x_1,\dots,x_n : r)$ be a torsion-free, one-relator group. Consider the following condition on the group G.

$(*_m)$ There exist exactly m homomorphisms $\alpha_1^G,\dots,\alpha_m^G$: $G \longrightarrow \{\pm 1\}$ defining G-module structures $Z_{\alpha_1^G},\dots,Z_{\alpha_m^G}$ on integers such that $H_2(G,Z_{\alpha_i^G}) = Z$ for $i = 1,\dots,m$. Applying elementary Nielsen transformations to the group $F = (x_1,\dots,x_n)$ we can obtain m presentations $N_F(r_i) \rightarrowtail F \xrightarrow{p_i} G$ of the group G such that each homomorphism α_i^G is induced by α_0, except maybe one which is induced by the trivial map ϵ, Note that this exceptional case takes place if and only if $r \in [F,F]$. For notational convenience we will assume that α_i^G is induced by ϵ if this is the case.

For any $m \times n$ matrix A with integer entries denote by $S(A)$ an isomorphism class of a cokernel of the homomorphism $Z^n \longrightarrow Z^m$ determined by A.

Now let us associate with a presentation $(x_1,\dots,x_n : r)$ of a torsion-free group G satisfying $(*_m)$ an unordered sequen ce $\{S(A^{\epsilon}_{r_1}),\ S(A^{\alpha_0}_{r_2}),\ldots,\ S(A^{\alpha_0}_{r_m})\}$ or $\{S(A^{\alpha_0}_{r_1}),\ldots,S(A^{\alpha_0}_{r_m})\}$ according as r belongs or not to $[F,F]$. Denote this sequence by $S(r)$ and put $S(r) = \emptyset$ if there are no homomorphisms $\alpha : G \longrightarrow \{\pm 1\}$ such that $H_2(G,Z_\alpha) = Z$.

<u>Theorem</u> A. $S(r)$ is an invariant of a torsion-free, one-relator group $G = (x_1,\ldots,x_n : r)$.

<u>Proof</u>. Let $H = (x_1,\ldots,x_n : s)$ be the group isomorphic to G and let $\varphi : H \longrightarrow G$ be the isomorphism. Plainly if G satisfies $(*_m)$ then so do H and homomorphisms $\alpha^G_1\varphi,\ldots,\alpha^G_m\varphi$ Denote $\alpha^G_i\varphi$ by α^H_i. From the naturality of cap product we have the following commutative diagrams

$$\begin{tikzcd}
H^1(G,Z) \arrow[r, "e^i_G \cap -"] \arrow[d, "\varphi^*"', "\cong"] & H_1(G,Z_{\alpha^G_i}) \arrow[d, "\varphi_*"', "\cong"] \\
H^1(H,Z) \arrow[r, "e^i_H \cap -"] & H_1(H,Z_{\alpha^H_i})
\end{tikzcd}$$

where $e^i_H \in H_2(H,Z_{\alpha^H_i}) = Z$ is the generator described in the previous section and $e^i_G = \varphi_*(e^i_H)$. Thus $S(e^i_G \cap -) = S(e^i_H \cap -)$ for $i = 1,\ldots,m$.

Now we can assume that also e^i_G are those generators described in section 3 (if this is not the case for some i one can consider presentation $(x_1,\ldots,x_n : \bar{r}_i)$ of G).

Assume that $r \in [F,F]$, then by 3.8(iii) we have $S(e^k_G \cap -) = S(A^{\alpha_0}_{r_i})$ and $S(e^i_H \cap -) = S(A^{\alpha_0}_{s_i})$ for $i = 1,\ldots,m$ hence the result. Now assume that $r \in [F,F]$. Since $S(e^i_G \cap -) = S(e^i_H \cap -)$ then 3.8(iv) implies $S(A^{\alpha_0}_{r_i}) = S(A^{\alpha_0}_{s_i})$ for $i > 1$.

If $i = 1$ then by 3.9 $S(A^{\epsilon}_{r_1}) = S(e^1_G \cap -)$ and $S(A^{\epsilon}_{s_1} -) = S(e^1_H \cap -)$ and our theorem is proved.

Modifications of a presentation in the definition of $S(r)$ were necessary for the proof of Theorem A. They are superflous in concrete calculations of $S\ r$ since we have

<u>Proposition</u> 4:1 Suppose given homomorphisms $\alpha, \beta : F \longrightarrow \{\pm 1\}$ and an automorphism φ of F such that $\beta\varphi = \alpha$. Then for any $r \in [F,F]_{\alpha}$, $S(A^{\alpha}_{r}) = S(A^{\beta}_{\varphi(r)})$.

<u>Proof</u>. Let M_{φ} be the $n \times n$ matrix with entries $(M_{\varphi})_{i,j} = \partial\varphi(x_i)/\partial x_j \in ZF$. Using the chain rule of differentiation $\partial\varphi(r)/\partial x_i = \sum_{j=1}^{n} \varphi(\partial r/\partial x_j)\partial\varphi(x_j)/\partial x_i$ [2] we can show that $A^{\beta}_{\varphi(r)} = \beta(M^T_{\varphi})A^{\alpha}_{r}\epsilon(M_{\varphi})$. The same formula allows us to show that for any two automorphisms φ_1, φ_2 of the group F we have $M_{\varphi_1\varphi_2} = \varphi_1(M_{\varphi_2})M_{\varphi_1}$. Therefore in order to prove our proposition it is sufficient to check that the matrix M_{φ} is invertible for any elementary transformation φ.

As was mentioned in the introduction Horadam has found invariants of torsion-free, one-relator groups with a relator in $[F,F]$. In our notation the invariant from [5] is nothing else than $S(A^{\epsilon}_{r})$. By means of examples we show that our invariant is stronger than that of Horadam.

<u>Example</u> 5.2. (see [5]) Let $G = (x,y : r = [x,y^4])$, $H = (x,y : s = [x^2,y^2])$. We have $S(A^{\epsilon}_{r}) = S(A^{\epsilon}_{s})$. On the other hand it is easy to see that the group G satisfies $(*_3)$ whereas H satisfies $(*_4)$. Thus $S(r) \neq S(s)$ ($S(r)$ is a sequence of length 3 whereas $S(s)$ is a sequence of length 4).

Example 4.3. Let $G = (x,y : r = [y,x^2][y,x^2]^x)$ and $H = (x,y : s = [y,x^2][y,x^2]^{\bar{x}^2})$. It is easy to check that both G and H satisfy (*3) and the corresponding homomorphisms α_i^G, α_i^H are induced by the following homomorphisms $\alpha_i : F = (x,y) \longrightarrow \{\pm 1\}$

$$\alpha_1(x) = +1, \quad \alpha_1(y) = +1,$$
$$\alpha_2(x) = -1, \quad \alpha_2(y) = +1$$
$$\alpha_3(x) = -1, \quad \alpha_3(y) = -1$$

Direct computations show that

$$A_r^{\ 1} = A_s^{\ 1} = \begin{pmatrix} 0 & -4 \\ +4 & 0 \end{pmatrix},$$

$$A_r^{\ 2} = A_r^{\ 3} = 0$$

$$A_s^{\ 2} = \begin{pmatrix} 0 & 0 \\ -4 & 0 \end{pmatrix}, \quad A_s^{\ 3} = \begin{pmatrix} +4 & 0 \\ -4 & 0 \end{pmatrix}.$$

Since α_1 is the trivial map ϵ, Horadam's invariant does not distinguish between these groups. On the second hand $S(r) \neq S(s)$ hence the groups G and H are nonisomorphic.

However the next example shows that our invariant is not sufficient to distinguish between all torsion-free, one-relator groups with a relator in $[F,F]_\alpha$ for some α.

Example 4.4. Let $G = (x,y : r = x^2y^{12})$, $H = (x,y : s = x^4y^6)$, then $S(r)$ Y $S(s)$, but G and H are nonisomorphic by the result of Meskin [9].

5. One-relator TPD-groups.

The following definition is a natural generalization of the definition of trivial Poincare duality group given by Ratcliffe [10].

Definition 5.1. We say that G is a trivial Poincare duality group of dimension n (TPDn-group) if there is a G-module structure Z_α on integers and a class $e \in H_n(G,Z_\alpha)$ such that for abelian group A on which G acts trivially the cap product with e induces an isomorphism $e \cap - : H^k(G,A) \longrightarrow H_{n-k}(G,Z_\alpha \otimes A)$ for every integer k (G acts on $Z_\alpha \otimes A$ diagonally). We say that G is an oriented TPDn-group if the trivial G-module Z satisfies the above condition (definition of Ratcliffe) otherwise we say that G is an unoriented TPDn-group.

Lemma 5.2. Let G be an FP-group (see [1] for the definition) and let $e \in H_n(G,Z_\alpha) = Z$ be a generator. Then G is TPDn-group if and only if $e \cap - : H^k(G,Z) \longrightarrow H_{n-k}(G,Z_\alpha)$ is an isomorphism for every integer k.

Proof. We split the proof into two cases.

Case 1. Let $A = \bigoplus_I Z$ be a free abelian group. Using arguments dual to those in the proof of lemma 2.2 in [1] one can show that there are commutative diagrams

$$\begin{tikzcd}
\bigoplus_I H^k(G,Z) \arrow[r, "\nu_k"] \arrow[d, "\bigoplus_I e\cap -"'] & H^k(G, \bigoplus_I Z) \arrow[d, "e\cap -"] \\
\bigoplus_I H_{n-k}(G,Z_\alpha) \arrow[r, "\mu_k"] & H_{n-k}(G, \bigoplus_I Z_\alpha \cong Z_\alpha \otimes (\bigoplus_I Z))
\end{tikzcd}$$

The homomorphisms μ_k are isomorphisms and ν_k are isomorphisms since G is FP-group, hence the result.

Case 2. Let A be an arbitrary abelian group. In this case we have an exact sequence $0 \longrightarrow K \longrightarrow F \longrightarrow A \longrightarrow 0$, in which K and F are free abelian groups. Using standard homological arguments we obtain the assertion in this general case.

By the definition a TPDn-group has trivial cohomological dimension equal to n. Since a one-relator group with torsion has an infinite trivial cohomological dimension we have

<u>Corollary</u> 5.3. A one-relator TPD-group is torsion-free.

As the cohomological dimension of a torsion-free one-relator group is at most two we have

<u>Corollary</u> 5.4. A one-relator group can not be TPDn-group for $n > 2$.

Finally looking on homology and cohomology of a torsion-free, one-relator group we obtain one more

<u>Corollary</u> 5.5. There are no unoriented one-relator TPD-groups of dimension 0 and 1.

Hence the following theorem gives us a full description of the class of unoriented one-relator TPD-groups.

<u>Theorem</u> B. A group $G = (x_1,\dots,x_n : r)$ is an unoriented TPD2-group if and only if $r \in [F,F]_\alpha$ for certain nontrivial homomorphism $\alpha : F \longrightarrow \{\pm 1\}$ and the matrix A_r^α is invertible.

<u>Proof</u>. Let G be an unoriented TPD2-group and let $\alpha : G \to \{\pm 1\}$ be the corresponding nontrivial homomorphism. By the definition $H_2(G,Z_\alpha) = Z$ and by 5.3 G is torsion-free. Hence by 3.1(i) $r \in [F,F]_\alpha$. Note that $r \notin [F,F]$ otherwise $H^1(G,Z) = Z^n$ and $H_1(G,Z_\alpha) = Z^{n-1}$ by 3.1(ii). If $\alpha = \alpha_0$ then by Proposition 3.8(iii) the matrix A_r^α is invertible. If $\alpha \neq \alpha_0$ let φ be an automorphism of F such that $\alpha\varphi = \alpha_0$. Note that $\varphi^{-1}(r) \in [F,F]_{\alpha_0}$ and $G = (x_1,\dots,x_n : \varphi^{-1}(r))$. By the previous case the matrix $A_{\varphi^{-1}(r)}^{\alpha_0}$ is invertible. Whence by Proposition 4.1 the matrix A_r^α is invertible.

Conversely assume that $r \in [F,F]_\alpha$ and the matrix A_r^α is invertible. First we show that the group G is torsion-free. Suppose then that $r = s^m$ for some $m > 1$.

Let $\alpha(s) = +1$. Since $\partial r/\partial x = (1 + s + \ldots + s^{m-1})\partial s/\partial x$ we have that $\widetilde{\alpha}(\partial r/\partial x) = m\widetilde{\alpha}(\partial s/\partial x)$. Thus $s \in [F,F]_\alpha$ by Lemma 1.1 and $A_r^\alpha = m.A_s^\alpha$ by Lemma 3.4, what contradicts invertibility of A_r^α.

Now let $\alpha(s) = -1$. From $\alpha(r) = +1$ it follows that m is even. Thus $r = t^2$ for some $t \in F$. As earlier we exclude the possibility $\alpha(t) = +1$. Assume then that $\alpha(t) = -1$. Since $\partial r/\partial x = (1 + t)\,\partial t/\partial x$ and $\partial^2 r/\partial x\partial y = \partial t/\partial y\,\epsilon(\partial t/\partial x) +$ $+ (1 + t)\partial^2 t/\partial x\partial y$ we conclude that $\widetilde{\alpha}(\partial^2 r/\partial x\partial y) = \widetilde{\alpha}(\partial t/\partial y)\epsilon(\partial t/\partial x)$ hence the matrix A_r^α is not invertible. This contradiction shows that the group G is torsion-free.

By Proposition 3.1(i) $H_2(G,Z_\alpha) = Z$. Let e_G^α be the generator of $H_2(G,Z_\alpha)$ described in Lemma 3.2. If $\alpha = \alpha_0$ then by Lemma 3.6 we have that $r \notin [F,F]$, hence our assertion follows from Proposition 3.8(iii). Otherwise we obtain it as in the first part of the proof.

Bibliography

1. R. Bieri, B. Eckmann, Finiteness Properties of Duality Groups, Comment. Math. Helv. 49(1974), 74-83.
2. R. Fox, Free differential calculus I, Ann. of Math. 57 (1953), 547-560.
3. G. Gromadzki, Homology of Groups with Twisted Coefficients, Thesis 1983 in polish.
4. P. Hilton, U. Stambach, A Course in Homological Algebra, Graduate Texts in Math., Berlin-Heidelberg-New York, Springer 1970.

5. K.J. Horadam, A quick test for nonisomorphism of one-relator groups, Proc. Amer. Math. Soc. 81 (1981), 195-200.

6. K.J. Horadam, One-Relator Groups and the Lower Central Series. II Invariants of One-Relator Groups, Math. Z. 179 (1982), 359-368.

7. R. Lyndon, Cohomology theory of groups with a single definition, Ann. of Math., 52(1950), 650-665.

8. Z. Marciniak, Poincare Duality Groups with One Defining Relation, Bull. Acad. Polon. Sci. 27 (1979), 27-31.

9. S. Meskin, The isomorphism problem for a class of one-relator groups, Math. Ann. 217 (1975), 53-57.

10. J.G. Ratcliffe, On One-Relator Groups which Satisfy Poincare Duality, Math. Z. 177 (1981), 425-438.

11. R. Fenn, D. Sjerve, Duality and cohomology for one-relator groups, Pacific J. Math. 103 (1982), 365-376.

Rebut el 5 de novembre del 1984

Institute of Mathematics,
Pedagogical University Bydgoszcz,
Chodkiewicza 30,
85-064 Bydgoszcz,
POLAND